The Chemical Evolution of Galaxies

[Tinsley 1980, *Fund. Cosmic Phys.*, **5**, 287]
[Rana 1991, *ARAA*, **29**, 129]

There a number of processes that affect the chemistry of galaxies. These include

• **Type II Supernovae:** In addition to creating iron-peak, and r-process elements, supernovae from high-mass stars produce the majority of a galaxy's α-process elements (oxygen, sulfur, neon, argon, etc.)

• **Type Ia Supernovae:** In the collapse of a white dwarf, virtually the entire star will be changed into iron-peak elements. It is generally assumed that most of the iron seen in a galaxy originated in these objects.

• **Novae:** Nova outbursts eject the products of CNO burning directly into space, hence they are a major contributer of nitrogen. Novae with high-mass white dwarfs can also produce a significant amount of neon, aluminum, magnesium, and sodium.

• **AGB stars:** Some mixing does occur in the thermal-pulsing phase of AGB stars. This will enhance the CNO abundances of a system (especially nitrogen) and can build-up s-process elements.

Because each these processes involves a different timescale, the elemental ratios (such as [O/Fe]) will be population dependent. Models for the build-up of these elements are more complex that those for a galaxy's photometric evolution. However, they can still be written in closed forms via a series of coupled differential equa- tions. Furthermore, with a few approximations, analytical solutions to chemical evolution can be found.

Metallicity Indicators

There are only a few ways to measure (or estimate) the metallicity of a a galaxy:

- Spectroscopy of Bright Stars:

Advantages:	Measures chemical history of ISM via stars
	Good for both Fe-peak and α-process elements
Disadvantages:	Requires high S/N absorption line spectra
	Only possible in very nearest galaxies
	Only possible for brightest (non-LTE) O-stars

- Spectroscopy of H II Regions:

Advantages:	Requires moderate S/N of emission-lines
	Sources are easy to find and measure
	Analysis is straightforward
Disadvantages:	Only measures present day ISM
	Only possible in star-forming systems
	Mostly limited to α-elements (best for O)
	Hard when abundances are super-solar

- Spectroscopy of Planetary Nebulae:

Advantages:	Requires moderate S/N of emission-lines
	Sources easy to find
	Analysis is straightforward
	Can be done in all galaxies
	Measures chemical history of ISM via stars
Disadvantages:	Much fainter than H II regions
	Mostly limited to α-elements (best for O)
	Oxygen possibly affected by stellar evolution
	Hard when abundances are super-solar

- Broadband Photometry of Red Giants:

Advantages:	Can be done in all galaxies
	Relatively easy (especially from space)
	Large numbers of stars available
Disadvantages:	Contrast is low; difficult to resolve the stars
	Crowding restricts data to low-density regions
	Does not yield data on individual elements
	Mostly limited to estimates of Fe-peak elements

- X-ray Spectroscopy of Hot Gas:

Advantages:	Can be done in distant galaxies/clusters
Disadvantages:	Limited (or no) spatial information
	Only measures present day (hot) ISM
	Easy for Fe; difficult for α-elements
	Interpretation can be difficult

- Absorption-Line Spectroscopy of Background Quasars:

Advantages:	Can be done in distant galaxies/clusters
	Can access large array of elements
Disadvantages:	Only measures one (halo) sight-line
	Interpretation may be difficult
	Only accesses one ionization state of element

- Spectroscopy of Integrated Light:

Advantages:	Can be done in distant galaxies/clusters
	Measures past ISM via stars
	Can measure both Fe-peak and α-elements
Disadvantages:	Requires high S/N aborption-line data
	Only produces luminosity-weighted mean value
	Age-metallicity degeneracy hard to break
	Results are model dependent

Modeling Chemical Evolution

To start, let's consider the types of parameters and variables that are involved. First, there are the global variables, all of which are a function of time.

M_g: Total mass of interstellar gas
M_s: Total mass of stars
M_w: Total mass of stellar remnants (white dwarfs)
M_t: Total mass of the system
E: the rate of mass ejection from stars
E_Z: the rate of metal ejection from stars
W: the creation rate of stellar remnants.

Naturally, $M_t = M_g + M_s + M_w$.

Next, there are the global parameters of the model, which the investigator specifies. Again, all can be a function of time.

Ψ: Rate of star formation
f: Rate of infall or outflow of material from the system
Z_f: Metal abundance of the infall (or outflow) material
$\phi(m)$: the Initial Mass Function

As in the equations of photometric evolution, the IMF should be normalized so that the total mass is one, as in (8.02).

Finally, you need four variables which come from stellar evolution

w: the mass of a stellar remnant
τ_m: the main-sequence lifetime of a star
m_{tn}: the turnoff mass of a population with $t = \tau$
p_z: the stellar recyclable mass fraction that is converted to metal z and then ejected into space.

Given the above variables and parameters, the goal is to derive $Z(t)$, the fraction of metals (individually, or as a group) in the interstellar medium as a function of time.

Equations of Chemical Evolution

There are five coupled differential equations which describe the chemical evolution of a system.

$$\frac{dM_t}{dt} = f \tag{9.01}$$

$$\frac{dM_s}{dt} = \Psi - E - W \tag{9.02}$$

$$\frac{dM_g}{dt} = -\Psi + E + f \tag{9.03}$$

$$\frac{dM_w}{dt} = W \tag{9.04}$$

$$\frac{d(ZM_g)}{dt} = -Z\Psi + E_Z + Z_f f \tag{9.05}$$

The equations are fairly simple to understand. Equation (9.01) is simple mass conservation. Equations (9.02), (9.03), and (9.04) keep track of the amount of mass that gets locked up in stars (or released into the ISM). Equation (9.05) is the most complex, as it describes how the metallicity of the interstellar medium changes with time. The first term of (9.05) refers to the amount of ISM metals that becomes locked up into stars, the second term gives the amount of metals being released by stars, and the third represents the amount of metals being brought in (or lost) from outside.

Although the list of variables mentioned above is formidable, not all the variables are independent. Consider E, the mass ejection rate from stars. Since mass loss only occurs during post-main sequence evolution, the rate of mass loss is related to the number of stars evolving off the main sequence at any time. If the galaxy

consisted only of a single population of stars, this rate would be given by

$$E(t) = N_{tn}(m_{tn} - w) = \mathcal{M}_0 \phi(m_{tn}) \frac{dm_{tn}}{dt}(m_{tn} - w) \qquad (8.32)$$

However, for a galaxy with on-going star formation, the calculation of mass ejection rate must count the main-sequence turnoff stars of all stellar ages. Thus the total amount of ISM returned from stars at time t is

$$E = \int_{m_{tn}}^{m_u} (m - w)\Psi(t - \tau_m)\phi(m, t - \tau_m)dm \qquad (9.06)$$

where m_u is the upper mass limit of the stellar IMF, and m_{tn}, the turnoff mass at time t. Similarly, the equation for the total mass of remnants formed is

$$W = \int_{m_{tn}}^{m_u} w\Psi(t - \tau_m)\,\phi(m, t - \tau_m)dm \qquad (9.07)$$

The equation for E_Z is a bit more complicated since it has two terms: one to represent the amount of *new* metals created by a star and released during mass loss, and a second to represent the amount of metals that were lost from the ISM when the star formed, but are now being re-released. Mathematically, this is

$$E_Z = \int_{m_{tn}}^{m_u} mp_z \Psi(t - \tau_m)\,\phi(m, t - \tau_m)dm + \int_{m_{tn}}^{m_u} (m - w - mp_z)Z(t - \tau_m)\Psi(t - \tau_m)\,\phi(m, t - \tau_m)dm \qquad (9.08)$$

Finally, there is an equation of metal conservation. If $\bar{Z}_s$ is the average metal content in stars, then the total amount of metals produced in a galaxy over a Hubble time is

$$\bar{Z}_s M_s + Z M_g = \int_0^t \int_{m_{tn}}^{m_u} mp_z \Psi(t' - \tau_m)\phi(m, t' - \tau_m)dt'dm \qquad (9.09)$$

Primary and Secondary Elements

The above equations assume that p_z, the fraction of a star which is converted into metals, is independent of the initial metallicity of the star. In other words, it assumes that the metal under consideration is a primary element. However, some elements can only be made if another element already exists. For example, to make nitrogen (via the CNO cycle), the star must already have some carbon. Thus, the mass ejection rate of a secondary element, X, is

$$\begin{aligned} E_X = &\int_{m_{tn}}^{m_u} m p_X Z(t-\tau_m)\Psi(t-\tau_m)\,\phi(m, t-\tau_m) dm + \\ &\int_{m_{tn}}^{m_u} (m - w - m p_X) X(t-\tau_m)\Psi(t-\tau_m)\,\phi(m, t-\tau_m) dm \end{aligned} \tag{9.10}$$

Note that this is similar to the equation for E_Z, in that it has two terms: the creation term and the recycle term. However, in this case, the creation term depends on the prior abundance of Z.

Analytic Approximation to Chemical Evolution

Obviously, solving the above coupled differential equations with their four free parameters (Ψ, ϕ, f, and Z_f) is a non-trivial numerical problem. However, the problem can be greatly simplified if you make two approximations.

The first approximation to make is to say that the initial mass function of stars is independent of time. That is, $\phi(m, t) = \phi(m)$. Since little is known about how the IMF changes as a function of galactic conditions, this may, or may not, be a good assumption. (The prevailing theory is that the IMF for a system with *no* metals is heavily biased towards extremely massive stars, but once the first bits of metals get introduced into the ISM, this bias goes away.)

The second approximation is call the *instantaneous recycling* approximation and it is a bit tricker. The approximation says that there are two types of stars in a galaxy: those that live forever, and those that evolve and die instantaneously. Although this sounds like a poor assumption, it's not as bad as it first appears. Recall that the timescales for stellar evolution:

Main Sequence Lifetimes

Spectral Type	Mass ($\mathcal{M}/\mathcal{M}_\odot$)	Luminosity ($\mathcal{L}/\mathcal{L}_\odot$)	Lifetime (years)
O5 V	60	7.9×10^5	5.5×10^5
B0 V	18	5.2×10^4	2.4×10^6
B5 V	6	820	5.2×10^7
A0 V	3	54	3.9×10^8
F0 V	1.5	6.5	1.8×10^9
G0 V	1.1	1.5	5.1×10^9
K0 V	0.8	0.42	1.4×10^{10}
M0 V	0.5	0.077	4.8×10^{10}
M5 V	0.2	0.011	1.4×10^{11}

Note the values. Stars with $\mathcal{M} > 5\mathcal{M}_\odot$ evolve in less than 10^8 years, which, in cosmological terms, is almost instantaneously. On the other hand, stars with mass less than about $1\mathcal{M}_\odot$ live forever. So the approximation only breaks down for a limited mass range.

Let's choose m_1 to be the dividing line between stars that live forever, and stars that evolve instantaneously. Let's also define three new quantities, the **Return fraction** of gas

$$R = \int_{m_1}^{\infty} (m - w)\phi(m)dm \tag{9.11}$$

the **Baryonic Dark Matter fraction**

$$D = \int_{m_1}^{\infty} w\phi(m)dm \tag{9.12}$$

and the **Net Yield** (of element i)

$$y_i = \frac{1}{1-R}\int_{m_1}^{\infty} mp_z\phi(m)dm \tag{9.13}$$

In words, R is the amount of mass a generation of stars puts back into the ISM, D is the amount of mass a generation of stars turns into stellar remnants, and y_i is the fraction of metal i produced by stars for every $1\mathcal{M}_\odot$ of material locked up into stars or remnants. The importance of these three quantities is that each depend only on the IMF. If we assume some universal form for the IMF, then R, D, and y_i are constants that depend only on stellar evolution. In other words, they are known quantities.

Now, let's take another look at the equations for E, W, and E_Z. If we assume $\phi(m)$ is independent of t and use the instantaneous recycling approximation, then

$$\begin{aligned} E &= \int_{m_{tn}}^{m_u} (m-w)\Psi(t-\tau_m)\phi(m,t-\tau_m)dm \\ &= \Psi(t)\int_{m_1}^{m_u}(m-w)\phi(m)dm = R\Psi \end{aligned} \tag{9.14}$$

Similarly, the equation for stellar remnants becomes

$$\begin{aligned} W &= \int_{m_{tn}}^{m_u} w\Psi(t-\tau_m)\phi(m,t-\tau_m)dm \\ &= \Psi(t)\int_{m_1}^{m_u} w\phi(m)dm = D\Psi \end{aligned} \tag{9.15}$$

and, after a bit of math,

$$E_Z = \Psi\left\{ZR + y_z(1-R)\right\} \tag{9.16}$$

With our two assumptions, the equations of chemical evolution become

$$\frac{d\mathcal{M}_t}{dt} = f \tag{9.17}$$

$$\frac{d\mathcal{M}_s}{dt} = (1 - R - D)\Psi \tag{9.18}$$

$$\frac{d\mathcal{M}_g}{dt} = -(1 - R)\Psi + f \tag{9.19}$$

$$\frac{d\mathcal{M}_w}{dt} = D\Psi \tag{9.20}$$

$$\frac{d(Z\mathcal{M}_g)}{dt} = -Z\Psi(1 - R) + y_z\Psi(1 - R) + Z_f f \tag{9.21}$$

Actually (9.21) can be further simplified by noting that

$$\frac{d(Z\mathcal{M}_g)}{dt} = Z\frac{d\mathcal{M}_g}{dt} + \mathcal{M}_g\frac{dZ}{dt} \tag{9.22}$$

Substituting (9.19) for dM_g/dt then yields

$$\mathcal{M}_g\frac{dZ}{dt} = y_z\Psi(1 - R) + (Z_f - Z)f \tag{9.23}$$

For secondary elements, the ejection rate from (9.10) becomes

$$\begin{aligned} E_X &= \Psi Z(1 - R)y_z + RX\Psi - y_z(1 - R)X\Psi \\ &= \Psi(1 - R)y_z(Z - X) + \Psi XR \end{aligned} \tag{9.24}$$

(In many cases, $X \ll Z$, so to simplify things, you can often get away with $Z - X \approx Z$.) Metal conservation for secondary elements is therefore

$$\frac{d(X\mathcal{M}_g)}{dt} = -X\Psi + \Psi(1 - R)(Z - X)y_x + RX\Psi + X_f f \tag{9.25}$$

But since

$$\frac{d(X\mathcal{M}_g)}{dt} = X\frac{d\mathcal{M}_g}{dt} + \mathcal{M}_g\frac{dX}{dt} = -X(1-R)\Psi + Xf + \mathcal{M}_g\frac{dX}{dt} \tag{9.26}$$

some of the terms cancel, so

$$\mathcal{M}_g\frac{dX}{dt} = \Psi(1-R)(Z-X)y_x + (X_f - X)f \tag{9.27}$$

Estimating the Stellar Yields

Models of stellar evolution and supernova nucleosynthesis can predict the amount of metals produced by a stellar population. But is there a way to empirically check the results of these (purely theoretical) calculations?

Let's start with the equation for the total amount of metals produced over a systems lifetime. Using the instantaneous recycling approximation,

$$\begin{aligned} \bar{Z}_s(\mathcal{M}_s + \mathcal{M}_w) + Z\mathcal{M}_g &= \int_0^t \int_{m_{tn}}^{m_u} m p_z \Psi(t' - \tau_m)\phi(m, t' - \tau_m) dt' dm \\ &= \int_0^t \Psi(t) dt \int_{m_1}^{\infty} m p_z \phi(m) dm \\ &= (1 - R)\, y_z \Psi_T \end{aligned} \tag{9.28}$$

where Ψ_T is the total amount of star formation over the history of the galaxy. Now that we've defined Ψ_T, let's get rid of it. If we integrate the equations for the amount of mass that gets locked up in metals and remnants,

$$\int_0^t \frac{d\mathcal{M}_s}{dt} = \int_0^t (1 - R - D)\Psi \Longrightarrow \mathcal{M}_s = (1 - R - D)\Psi_T \tag{9.29}$$

$$\int_0^t \frac{d\mathcal{M}_w}{dt} = D\Psi \Longrightarrow \mathcal{M}_w = D\,\Psi_T \tag{9.30}$$

So

$$\begin{aligned} \bar{Z}_s &= \frac{(1 - R) y_z \Psi_T}{(1 - R)\Psi_T} - \frac{\mathcal{M}_g Z}{\mathcal{M}_s + \mathcal{M}_w} \\ &= y_z - \frac{\mathcal{M}_g Z}{\mathcal{M}_s + \mathcal{M}_w} \end{aligned} \tag{9.32}$$

Now let's define μ as the gas fraction of a galaxy

$$\mu = \frac{\mathcal{M}_g}{\mathcal{M}_g + \mathcal{M}_s + \mathcal{M}_w} = \frac{\mathcal{M}_g}{\mathcal{M}_t} \tag{9.32}$$

So

$$\bar{Z}_s = y_z - \frac{\mathcal{M}_t \mu}{\mathcal{M}_t - \mathcal{M}_t \mu} Z = y_z - \left(\frac{\mu}{1-\mu} \right) Z \tag{9.33}$$

So, if you choose a region (say, the solar neighborhood), and measure the average stellar metalllicity, the present metal abundance in the interstellar medium, and the gas fraction, you can estimate the stellar yield of the elements in question.

Probing Star Formation

The above equations allow us to estimate the history of star formation in the solar neighborhood. For example, if we re-write (9.18) as

$$\Psi = \frac{1}{1 - R - D} \frac{dM_s}{dt} \tag{9.34}$$

and split the derivative into three parts, we get

$$\Psi = \frac{1}{1 - R - D} \left(\frac{d \log \mathcal{M}_s}{d \log Z} \right) \left(\frac{d \log Z}{dt} \right) \left(\frac{d\mathcal{M}_s}{d \log \mathcal{M}_s} \right) \tag{9.35}$$

Now consider the derivatives. Since we can measure the metallicity of solar neighborhood stars, we can determining how much stellar mass there is as a function of metallicity. Thus the first derivative in (9.35) is a measureable quantity. Similarly, if we study nearby F-stars, and compare their absolute luminosities to the F-star zero-age main sequence luminosity, we can estimate how much main-sequence evolution has occurred, *i.e.*, we can estimate their ages. If we measure the stars' metallicities as well, then we have $d \log Z/dt$. Finally, the last derivative is simply $\ln(10)\mathcal{M}_s$. Thus, we can measure the history of star formation in the solar neighborhood.

This leads to the long-standing G-dwarf problem. Simple models of chemical evolution (say, a closed-box model or a constant in-fall model) predict many more low-metallicity stars in the solar neighborhood than are observed.

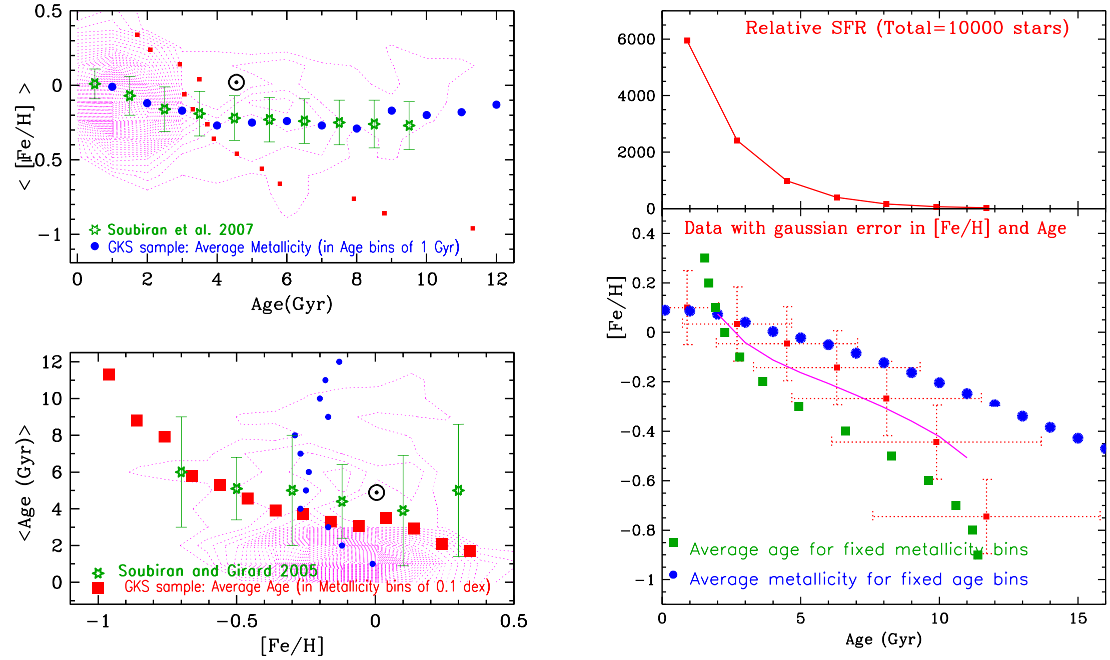
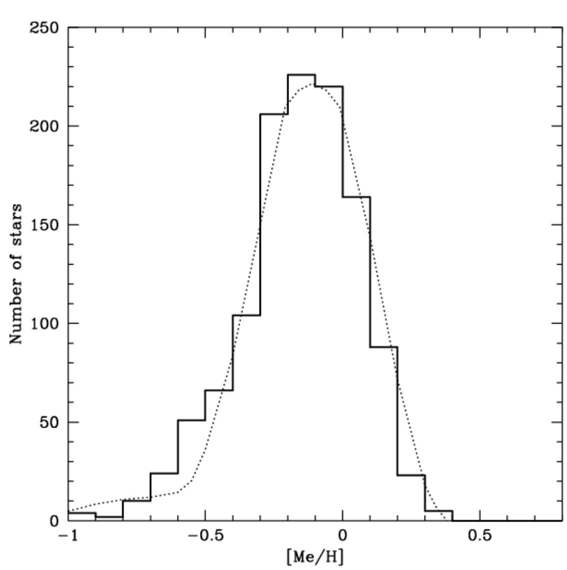
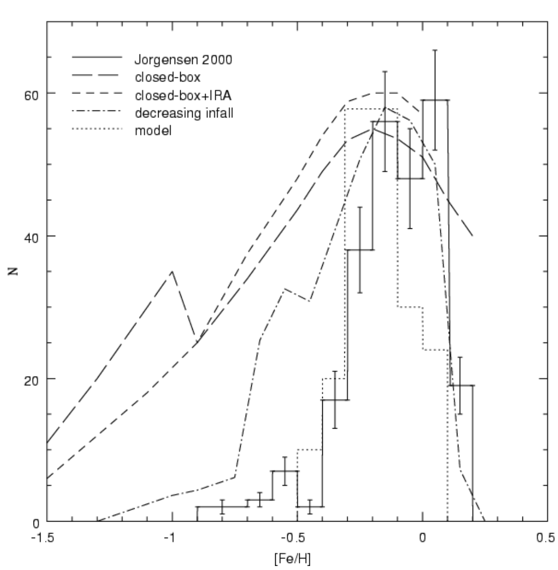


Figure 4. Left: Local Age-Metallicity relationship, in 1 Gyr age bins (*top*) and in 0.1 dex metallicity bins (*bottom*); *isocontours* correspond to the data of GKS (Nordstrom et al. 2004), *large symbols* show the corresponding relationships of age vs mean metallicity (*top*) and Metallicity vs mean age (*bottom*), *small symbols* the complementary relationship, and data with error bars designate results of independant studies with different samples (from Subiran et al. 2007 and Subiran and Girard 2005). Notice that in both panels the Sun appears to be slightly metal-rich for its age or slightly older than stars of the same metallicity. **Right:** By using simulated data as input for the AMR (points with 1-σ Gaussian error bars in lower panel, as indicated by *dotted lines*) and a star formation rate exponentially increasing with time (top panel), one may derive the corresponding age vs average metallicity (*circles*) and metallicity vs average age (*squares*) relations; the former is flatter than and the latter is steeper than the input relationship (as with the case of the real data, see left pannels). Their average (*solid curve*) is a better approximation to the input data.

[Prantzos 2008]



[Nordström *et al.* 2004]



[Casuso *et al.* 2004]

To get the history of matter infall, we can do a similar manipulation with equation (9.19)

$$
\begin{aligned}
f &= \frac{d\mathcal{M}_g}{dt} + (1-R)\Psi \\
&= \frac{d\mathcal{M}_g}{d\log Z} + \left\{ \frac{1-R}{1-R-D} \frac{d\mathcal{M}_s}{d\log Z} \right\} \frac{d\log Z}{dt} \qquad (9.36)
\end{aligned}
$$

If we measure the metallicities of clouds of H I and H_2 of different masses, then all terms of this equation are known, and the history of matter infall can be found.

The Closed Box Model of Chemical Evolution

As an example of what a chemical evolution model can do, consider a closed system, where all the material for current star formation comes from mass lost by a previous generation of stars. In this case, there is no infall, and, from (9.23),

$$\mathcal{M}_g \frac{dZ}{dt} = y_z \Psi(1-R) + (Z_f - Z) f = y_z \Psi(1-R) \tag{9.37}$$

In addition, from (9.19), we have

$$\frac{d\mathcal{M}_g}{dt} = -(1-R)\Psi + f = -(1-R)\Psi \tag{9.38}$$

By dividing these two equations, we get

$$\mathcal{M}_g \frac{dZ}{dt} \bigg/ \frac{d\mathcal{M}_g}{dt} = \mathcal{M}_g \frac{dZ}{d\mathcal{M}_g} = -y_z \tag{9.39}$$

Since y_z is a constant of stellar evolution

$$\int_{Z_0}^{Z_1} dZ = -y_z \int_{\mathcal{M}_{g_0}}^{\mathcal{M}_{g_1}} \frac{d\mathcal{M}_g}{\mathcal{M}_g} \Longrightarrow Z_1 - Z_0 = -y_z \ln\left(\frac{\mathcal{M}_{g_0}}{\mathcal{M}_{g_1}}\right) \tag{9.40}$$

where Z_0 and $\mathcal{M}_{g_0}$ represent the initial metallicity and gas mass of the galaxy, and Z_1 and $\mathcal{M}_{g_1}$ represent those quantities today. Now let

$$\mu = \left(\frac{\mathcal{M}_g}{\mathcal{M}_t}\right) \quad \sigma = \left(\frac{\mathcal{M}_s}{\mathcal{M}_t}\right) \quad \delta = \left(\frac{\mathcal{M}_D}{\mathcal{M}_t}\right) \tag{9.41}$$

so

$$Z_1 - Z_0 = -y_z \ln\left(\frac{\mu_1}{\mu_0}\right) \tag{9.42}$$

or

$$\mu_1 = \mu_0 \exp\left\{-\frac{Z_1 - Z_0}{y_z}\right\} \tag{9.43}$$

In other words, as the system evolves, the gas fraction will decrease exponentially with Z. Of course, we can't measure the metallicity evolution of the ISM directly, but we can use stars as a probe. If we take the derivative of (9.43) with respect to Z, then

$$\frac{d\mu}{dZ} = -\frac{\mu_1}{y_z} \exp\left\{-\frac{Z_1 - Z_0}{y_z}\right\} \tag{9.44}$$

Meanwhile, through(9.18) and (9.19)

$$\frac{d\mathcal{M}_s}{dt} \Bigg/ \frac{d\mathcal{M}_g}{dt} = \frac{d\sigma}{d\mu} = -\frac{(1 - R - D)}{(1 - R)} \tag{9.45}$$

so

$$\frac{d\sigma}{dZ} = \left(\frac{d\mu}{dZ}\right)\left(\frac{d\sigma}{d\mu}\right) = \left(\frac{\mu}{y_z}\right)\left(\frac{1 - R - D}{1 - R}\right) \exp\left\{-\frac{Z_1 - Z_0}{y_z}\right\} \tag{9.46}$$

Finally, if we put this equation in terms of $\log Z$, instead of Z, then

$$\frac{d\sigma/\sigma_1}{d\log Z} = (\ln 10)\left(\frac{Z_0}{y_z}\right)\left(\frac{1 - R - D}{1 - R}\right)\left(\frac{\mu_1}{\sigma_1}\right) \exp\left\{-\frac{Z_1 - Z_0}{y_z}\right\} \tag{9.47}$$

Because the number of stellar metallicity measurements has (traditionally) not been extremely large, many times people plot the cumulative distribution, *i.e.,* the number of stars with metallicities less than Z. This is simply found by integrating (9.47). If we collect the constant terms and let

$$G = \left(\frac{1 - R - D}{1 - R}\right)\left(\frac{\mu_1}{\sigma_1}\right) \tag{9.48}$$

then for the cumulative distribution

$$\frac{\sigma}{\sigma_1} = 1 - G\left\{\exp\left(-\frac{Z_1 - Z_0}{y_z}\right) - 1\right\} \tag{9.49}$$

When this is fit against observations, it is clear that the metallicities of stars in the solar neighborhood cannot be fit with a closed box model of galactic evolution. Either $f \neq 0$, or $Z_0 \neq 0$, or the initial population of stars did not have the same IMF as the stars today, or there are severe chemical inhomogeneities in the ISM, and star formation occurs preferentially in regions with high metallicity.

A similar closed-box calculation can be performed for secondary elements. For these, if you divide (9.27) by (9.19), then

$$\mathcal{M}_g \frac{dX}{dt} \Big/ \frac{d\mathcal{M}_g}{dt} = \mathcal{M}_g \frac{dX}{d\mathcal{M}_g} = \frac{\Psi(1-R)(Z-X)y_x}{-\Psi(1-R)} \tag{9.50}$$

If we assume that $X \ll Z$, then this simply reduces to

$$\mathcal{M}_g dX = -y_z Z d\mathcal{M}_g \tag{9.51}$$

Now if we assume that $\mathcal{M}_g = \mathcal{M}_t$ at $t = 0$, then from (9.42)

$$\mathcal{M}_g = \mathcal{M}_t \exp\left\{-\frac{Z_1 - Z_0}{y_z}\right\} \tag{9.52}$$

or

$$d\mathcal{M}_g = -\left(\frac{\mathcal{M}_g}{y_z}\right)\exp\left\{-\frac{Z_1 - Z_0}{y_z}\right\} dZ = \left(\frac{\mathcal{M}_g}{y_z}\right) dZ \tag{9.53}$$

Thus

$$\mathcal{M}_g dX = \frac{y_x}{y_z} Z dZ \Longrightarrow X = \frac{1}{2}\left(\frac{y_x}{y_z}\right) Z^2 \tag{9.54}$$

In other words, if X is a secondary element, then a plot of log-X versus log-Z (*i.e.*, [X] vs. [Z]) should have a slope of 2 (and presumably go through the solar value).

Timescales for Chemical Evolution

The timescales for chemical evolution are simple to derive. First, let's consider the gas consumption timescale. In the absense of accretion, how long does it take to use up the gas?

$$\tau_* = \mathcal{M}_g \Bigg/ \left| \frac{d\mathcal{M}_g}{dt} \right| = \frac{\mathcal{M}_g}{-(1-R)\Psi + f} = \frac{\mathcal{M}_g}{|(1-R)\Psi|} \tag{9.55}$$

In the solar neighborhood, $\mathcal{M}_g \sim 5.7\mathcal{M}_\odot$ pc^{-2}, and the star formation rate is $\Psi \sim 4.2\mathcal{M}_\odot$ pc^{-2} Gyr^{-1}. For $R \sim 0.4$, that means that the gas should e-fold in $\tau_* \sim 2$ Gyr, much less than a Hubble time.

Similarly, we can calculate the timescale for chemical enrichment.

$$\begin{aligned}\tau_z = Z \Bigg/ \left| \frac{dZ}{dt} \right| &= \frac{Z}{|y_z\Psi(1-R) + (Z_f - Z)f| / \mathcal{M}_g} \\ &= \frac{\mathcal{M}_g Z}{|\Psi(1-R)y_z|} = \frac{\tau_* Z}{y_z} \end{aligned} \tag{9.56}$$

Since $y_z \sim Z$, $\tau_z \sim \tau_*$. In other words, for the solar neighborhood, the metallicity (and amount) of the gas should e-fold rather quickly. This strongly suggests that some our region of space has received matter from some other region.

The Balanced Infall Model

A closed-box model is not realistic for most systems: there is good evidence that infall of new material plays a part in the chemical evolution of a region. So let's calculate (for simplicity) a model where the infall just balances the amount of material becoming locked up into stars. In other words, a model where the mass in the interstellar medium stays constant. In this case

$$\frac{d\mathcal{M}_g}{dt} = -\Psi + E + f = -(1-R)\Psi + f = 0 \tag{9.57}$$

So

$$\Psi = \frac{f}{1-R} \tag{9.58}$$

The equation for metals is then

$$\mathcal{M}_g \frac{dZ}{dt} = \Psi(1-R)y_z + (Z_f - Z)f = f(y_z + Z_f - Z) \tag{9.59}$$

Also, since there is accretion

$$\frac{d\mathcal{M}_t}{dt} = f \tag{9.60}$$

So

$$\mathcal{M}_g \frac{dZ}{dt} \bigg/ \frac{d\mathcal{M}_t}{dt} = \mathcal{M}_g \frac{dZ}{d\mathcal{M}_t} = \frac{f(y_z + Z_f - Z)}{f} = y_z + Z_f - Z \tag{9.61}$$

If we integrate this

$$\int_{Z_0}^{Z_1} \frac{dZ}{y_z + Z_f - Z} = \int_{\mathcal{M}_0}^{\mathcal{M}} \frac{d\mathcal{M}_t}{\mathcal{M}_g} \tag{9.62}$$

then

$$\ln\left\{\frac{y_z + Z_f - Z_0}{y_z + Z_f - Z_1}\right\} = \frac{\mathcal{M} - \mathcal{M}_0}{\mathcal{M}_g} \tag{9.63}$$

Now let ν represent the total amount of mass accreted, scaled to the mass in the ISM, *i.e.*, $\nu = (\mathcal{M} - \mathcal{M}_0)/\mathcal{M}_g$. Then

$$Z_1 = (y_z + Z_f)(1 - e^{-\nu}) + Z_0 e^{-\nu} \tag{9.64}$$

If the galaxy began with $Z_0 \sim 0$ and $Z_f \sim 0$, then

$$Z_1 = y_z(1 - e^{-\nu}) \tag{9.65}$$

Note that if you also assume that the galaxy began as an entirely gaseous system, then

$$\nu = \frac{\mathcal{M} - \mathcal{M}_0}{\mathcal{M}_g} = \frac{\mathcal{M} - \mathcal{M}_0}{\mathcal{M}_0} = \mu^{-1} - 1 \tag{9.66}$$

Thus, as $\mu \rightarrow 0$, $\nu \rightarrow \infty$, and $Z_1 \rightarrow y_z$. In other words, the metallicity of the system asymptotes out at the value of the stellar yield.

The balanced infall model is similar to the closed-box model in that $Z \propto y_z$, so the metallicity of a system is actually measuring the stellar yields. Also, note that Z/y_z is not a strong function of the gas fraction, or Ψ/f. It depends mostly on the current properties of the system.

The Mass-Metallicity Relation for Galaxies

The chemical evolution formulism described above provides a context to explain the correlation between luminosity (mass) and metallicity seen in galaxies. For simplicity, let's consider an elliptical galaxy. As you recall, there is a correlation between the absolute magnitude of ellipticals and their color, with large bright galaxies being redder. This is often interpreted as a metallicity effect.

From (9.09), the total mass of metals formed through the history of a galaxy is

$$Z\mathcal{M}_t = \int_0^t \int_{m_{t'}}^{\infty} m p_z \Psi(t' - \tau_m) \phi(m, t' - \tau_m) dt' dm \qquad (9.67)$$

which, in the instantaneous recycling approximation, is

$$Z\mathcal{M}_t = \int_0^t \Psi(t) dt \int_{m_1}^{\infty} m p_z \phi(m) dm \qquad (9.68)$$

From (9.28) and (9.13), this is

$$Z\mathcal{M}_t = \frac{\mathcal{M}_s}{1 - R - D} \cdot y_z (1 - R) = y_z \mathcal{M}_s \frac{1 - R}{1 - R - D} \qquad (9.69)$$

In the early stages of galaxy formation $\mathcal{M}_g \sim \mathcal{M}_t$, and $D \sim 0$, so

$$Z \sim \frac{y_z \mathcal{M}_s}{\mathcal{M}_t} \qquad (9.70)$$

Now we can use two possible arguments. First, let's suppose that the supernova rate early in the galaxy's history was high, and that these explosions pumped enough energy into the interstellar

medium to cause the gas to escape. Since the number of supernovae is proportional to the number of stars in the galaxy

$$\frac{G\mathcal{M}_t\mathcal{M}_g}{R} \sim \frac{G\mathcal{M}_t^2}{R} \sim E_{SN} \propto \mathcal{M}_s \Longrightarrow \frac{\mathcal{M}_t^2}{R} \propto \mathcal{M}_s \tag{9.71}$$

Now, if we assume some mass-radius relation, $\mathcal{M}_t \propto R^\alpha$, then

$$\frac{\mathcal{M}_t^2}{R} \propto \mathcal{M}_t^{2-(1/\alpha)} \propto \mathcal{M}_s \tag{9.72}$$

But from (9.70), $\mathcal{M}_t \propto (\mathcal{M}_s/Z)$, so

$$\left(\frac{\mathcal{M}_s}{Z}\right)^{2-(1/\alpha)} \propto \mathcal{M}_s \Longrightarrow Z \propto \mathcal{M}^{\frac{\alpha-1}{2\alpha-1}} \tag{9.73}$$

To get the (very rough) empirical value of $Z \sim \mathcal{M}_s^{0.25}$, $\alpha \sim 1.5$.

Alternatively, suppose that elliptical galaxies were formed during proto-galaxy collisions. If the efficiency of star formation depends on the mass of the system

$$\left(\frac{\Psi}{\mathcal{M}_g}\right) \propto \mathcal{M}_t^p \Longrightarrow \left(\frac{\mathcal{M}_s}{\mathcal{M}_g}\right) \propto \mathcal{M}_t\mathcal{M}_t^p \Longrightarrow \mathcal{M}_s \propto \mathcal{M}^{p+1} \tag{9.74}$$

In the above proportionalities, we've assumed that early on, $\mathcal{M}_g \sim \mathcal{M}_t$, and the mass of stars we see today, $\mathcal{M}_s$, is proportional to the original star formation rate. With this relation, and (9.70), we get

$$\mathcal{M}_s \propto \mathcal{M}_t^{p+1} \propto \left(\frac{\mathcal{M}_s}{Z}\right)^{p+1} \Longrightarrow Z \propto \mathcal{M}_s^{\frac{p}{p+1}} \tag{9.75}$$

and $p \sim 1/3$ recovers the empirical relation. Note that in both cases, $Z_s \rightarrow y_z$, so eventually the mass-metallicity relation has to flatten out.